Nilpotent Algebras Generated by Two Units, i and j, Such That i^2 Is Not an Independent Unit.*

BY GUY WATSON SMITH.

I. *Introduction.*

The problem of referring all hypercomplex number systems to a relatively small number of typical forms was suggested by Hamilton,† but with the exception of De Morgan's discussion of double and triple algebras, nothing much was done till Benjamin Peirce ‡ worked out all algebras of deficiency zero and one. Starkweather § worked out algebras of deficiency two. He showed also that algebras of n units could be obtained from those of $n-1$ units. Cartan ‖ using the characteristic equation developed the semi-simple and the nilpotent sub-algebras, and showed the possibility of representing every algebra by means of units with double character. Taber ¶ reestablished the results of Peirce and extended them to any domain of rationality for the coordinates. Wedderburn ** and Voghera †† made an advance in the treatment of the hypercomplex algebra by basing their work on the conception of invariant classes of numbers in the algebra.

Besides this direct line of development there have been two others. The first is by means of the continuous group, the second is by using the matrix

* This paper considers only linear associative algebras whose coordinates are taken from the field of ordinary complex numbers.

† "Lectures on Quaternions," Preface, pp, 29–31.

‡ "Linear Associative Algebra," AMERICAN JOURNAL OF MATHEMATICS, Vol. IV (1881) pp. 97–192.

§ "Non-Quaternion Number Systems Containing no Skew Units," AMERICAN JOURNAL OF MATHEMATICS, Vol. XXI (1899), pp. 369–386.

‖ "Les groupes bilineaires et les systemes de nombres complexes, *Annales de la Facultedes Sciences de Toulouse,* Vol. XII (1898), B. pp. 1–99.

¶ "On Hypercomplex Number Systems," *Am. Math. Soc. Trans.*, Vol. V (1904), pp. 509–548.

** "On Hypercomplex Numbers," *London Math. Soc. Proceedings,* Series 2, Vol. VI (1907), pp. 77–118.

†† "Zusammenstellung der Irreduziblen Komplexen Zahlensysteme in sechs Einheiten," *Denkschriften der Math. Nat. Klasse der Kais. Akad. der Wiss.* Wien., Vol. LXXXIV (1908), pp. 269–329.

theory. The first method was used by Scheffers,* Molien,† and Study,‡ the second by Shaw § and Frobenius. ||

Shaw regards all associative numbers as belonging to an associative algebra of an infinite number of units, the "associative units," λ_{rst}, which are elementary matrices. Each associative number is a linear combination of these units, so the theory of linear associative algebra is the theory of these associative units. He shows that the presence of a modulus is not necessary, thus making the methods particularly applicable to nilpotent systems. He proves that the equation of an algebra determines all the units but those which form a nilpotent system, and consequently to get all linear associative algebras we must first determine all nilpotent algebras. Benjamin Peirce ¶ was the first to recognize the importance of nilpotent algebras. Furthermore, algebras of order n may be found without first knowing those of order $n-1$. By selecting a base and adjoining to this a nilpotent unit, an ever-increasing system of nilpotent algebras may be determined.

The simplest case of the Shaw canonical form ** of a nilpotent algebra is that in which there is only a single generator, j, whose μ_1-th power vanishes,

$$j,\ j^2,\ j^3,\ j^4,\ \ldots,\ j^{\mu_1-1}.$$

Three lemmas concerning polynomials in this nilpotent have been introduced for use in the handling of the next simplest case, namely, that in which there are two generators, j and another nilpotent unit, i, whose square is not an independent unit in the canonical form. The unit j is such that $j^{\mu_1-1}\neq 0$, $j^{\mu_1}=0$ and $ij^{\mu_2-1}\neq 0$, $ij^{\mu_2}=0$, where μ_1 and μ_2 are multiplicities of j relative to j and to i respectively, and $\mu_1 \geqq \mu_2$. The expressions i, j^s, ij^t where $0<s<\mu_1$, and $0<t<\mu_2$, are the independent units of the system. The algebra is therefore of order $\mu_1+\mu_2-1$, or if we insert a modulus η, of order $\mu_1+\mu_2$. The μ_1 power of every number must vanish, hence the deficiency in this case is μ_2. There is at least one hypernumber which does not vanish for a power lower than μ_1,

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* "Zurückführung complexer Zahlensysteme auf typische Formen," *Math. Annalen*, Vol. XXXIX (1891), pp. 293–390.

† "Ueber Systeme höherer complexen Zahlen," *Math. Annalen*, Vol. XLI (1893) pp. 83–156.

‡ "Ueber Systeme von complexen Zahlen," *Gött. Nach.* (1889), pp. 237–268. "Complexe Zahlen und Transformationsgruppen," *Leipzig Berichte*, Vol. XLI (1889), pp. 177–228.

§ "Theory of Linear Associative Algebra," *Am. Math. Soc. Trans.*, Vol. IV (1903) pp. 251–287. "On Nilpotent Algebras," *Am. Math. Soc. Trans.*, Vol. IV (1903), pp. 405–422.

|| "Theorie der hypercomplexen Grossen," *Berlin Berichte* (1903), pp. 504–537, 634–645.

¶ *Loc. cit.*, p. 118.

** *Loc. cit.*, p. 406.

and there may be several such. The base unit i is chosen such that $ij^{\mu_2}=0$, $ij^{\mu_2-1}\neq 0$, the various products ij, ij^2, ..., do not contain terms in j alone, such as aj^m, and μ_2 is such that there is no hypernumber i' satisfying the conditions on i for which $i'j^{\mu_3}=0$, $i'j^{\mu_3-1}\neq 0$, $\mu_3>\mu_2$. In other words, i is not the product of any hypernumber into j, and we do not have for any power of j, $ij^s=bj^t$. These characters of i and j are essential. The two sequences $j, j^2, \ldots, j^{\mu_1-1}$ and $i, ij, ij^2, \ldots, ij^{\mu_2-1}$ constitute the two shear regions of j. In the cases Peirce considered, that of deficiency zero contains no i; that of deficiency unity must have $\mu_2=1$, hence $ij=0$. In Starkweather's types where only two generators enter, $\mu_2=2$, hence ij is a unit but $ij^2=0$. In the types herein considered the deficiency does not play any rôle at all. The investigation is along the line of Shaw's construction by *generators* and not by classification by numerical invariants, other than those entering the equations of condition. The associative units are used, though not indispensable, for convenience in making reductions.

II. *Lemmas Concerning Polynomials of a Single Nilpotent j.*

Lemma I. *If j is a nilpotent number ($j^{\mu_1}=0$), and if $F(j)$ and $G(j)$ are polynomials in j, a quotient $Q(j)$ can always be obtained $\dfrac{F(j)}{G(j)}$ provided $F(j)$ does not contain a term of lower order than does $G(j)$.*

Proof: The lemma will be proved if we can find a polynomial, $Q(j)$ of not more than μ_1 terms such that $F(j)=Q(j)\cdot G(j)$, or

$$f_0+f_1j+f_2j^2+\ldots\ldots=(q_0+q_1j+q_2j^2+\ldots\ldots)(g_0+g_1j+g_2j^2+\ldots\ldots).$$

Equating coefficients of corresponding powers of j on both sides since they belong to independent units,

$$f_0=q_0g_0,\quad f_1=q_0g_1+q_1g_0,\quad f_2=q_0g_2+q_1g_1+q_2g_0,\quad \ldots\ldots\ldots\ldots$$

These equations can not be solved if $F(j)$ starts with a lower power of j than $G(j)$, that is, if

$$f_0=f_1=f_2=\ldots\ldots=f_h=0,\ f_{h+1}\neq 0,$$

while $g_0=g_1=\ldots\ldots=g_{k+h}=0$ $k>0$. If, however, $f_0=f_1=\ldots\ldots=f_h=0$, $f_{h+1}\neq 0$, while $g_0=g_1=\ldots\ldots=g_{h-k}=0$, $g_{h-k+1}\neq 0$, $0<k<h+1$, then the first $h-k+1$ of the above equations are identically satisfied, the next k equations give $q_0=q_1=\ldots\ldots=q_{k-1}=0$ and from the remaining μ_1-h-1 equations, q_k, q_{k+1}, ..., $q_{\mu_1-h+k-1}$ may be uniquely determined, leaving q_{μ_1-h+k}, $q_{\mu_1-h+k+1}$, ..., q_{μ_1-1} arbitrary.

LEMMA II. *Consider the equation $L(j)\cdot M(j)=0$ where*
$L(j)=l_0+l_1j+l_2j^2+\ldots\ldots+l_{\mu_1-1}j^{\mu_1-1}$, $M(j)=m_0+m_1j+m_2j^2+\ldots\ldots+m_{\mu_1-1}j^{\mu_1-1}$.

(a). If $l_0\neq 0$, $M(j)$ vanishes entirely.

PROOF: $(l_0+l_1j+l_2j^2+\ldots\ldots+l_{\mu_1-1}j^{\mu_1-1})(m_0+m_1j+\ldots\ldots+m_{\mu_1-1}j^{\mu_1-1})=0$. Then

$$l_0m_0=0\ \therefore\ m_0=0,\quad l_0m_1+l_1m_0=0\ \therefore\ m_1=0,\quad l_0m_2+l_1m_1+l_2m_2=0\ \therefore\ m_2=0,$$
$$\ldots\ldots\ldots\ldots\ldots,\quad l_0m_{\mu_1-1}+\text{ vanishing terms }=0\ \therefore\ m_{\mu_1-1}=0.$$

(b). If $l_0=l_1=l_2=\ldots\ldots=l_n=0$, $l_{n+1}\neq 0$, $n<\mu_1-1$, then $L=j^{n+1}L_{n+1}(j)$, $M(j)j^{n+1}L_{n+1}(j)=0$, where $L_{n+1}(j)=l_{n+1}+l_{n+2}j+\ldots\ldots+l_{\mu_1-1}j^{\mu_1-n-2}$ and $l_{n+1}\neq 0$. Applying (a) to the equation $M(j)j^{n+1}\cdot L_{n+1}(j)=0$ gives

$$M(j)j^{n+1}=m_0j^{n+1}+m_1j^{n+2}+\ldots\ldots+m_{\mu_1-n-2}j^{\mu_1-1}=0,$$
$$\therefore\ m_0=m_1=m_2=\ldots\ldots=m_{\mu_1-n-2}=0\ \text{ or }\ M(j)=j^{\mu_1-n-1}M_1(j),$$

where $M_1(j)=m_{10}+m_{11}j+m_{12}j^2+\ldots\ldots+m_{1n}j^n+$ any terms whatever up to j^{μ_1-1}.

LEMMA III. *If $C(j)$ is a polynomial in j with non-vanishing constant term, then j can be expressed as a polynomial in (jC).*

PROOF: Let $C(j)=a_0+a_1j+a_2j^2+\ldots\ldots+a_{\mu_1-1}j^{\mu_1-1}$, $a_0\neq 0$, then

$$jC=a_0j+a_1j^2+a_2j^3+\ldots\ldots+a_{\mu_1-2}j^{\mu_1-1}:$$

$$(jC)^2=a_0^2j^2+2a_0a_1j^3+(2a_0a_2+a_1^2)j^4+\ldots\ldots,\qquad (jC)^3=a_0^3j^3+3a_0^2a_1j^4+\ldots\ldots,$$
$$(jC)^4=a_0^4j^4+\ldots\ldots,\qquad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots$$

And since $a_0, a_0^2, a_0^3, \ldots\ldots, a_0^{\mu_1-1}\neq 0$ by hypothesis the determinant of coefficients does not vanish, and we may solve these equations for $j, j^2, j^3, \ldots\ldots, j^{\mu_1-1}$ in terms of powers of (jC).

III. *Expressions for the Generators.*

From a nilpotent system choose any nilpotent expression j to be a unit, called the adjunct unit, such that j is a number which has as high a non-vanishing power as any number of the system, and a set of expressions (in this paper one, say i), called the base. Then it is known that any hypernumber of the system is linearly expressible in terms of the $\mu_1+\mu_2-1$ independent units

$$i,\ j,\ ij,\ j^2,\ ij^2,\ j^3,\ ij^3,\ \ldots\ldots,\ ij^{\mu_2-1},\ j^{\mu_2},\ j^{\mu_2+1},\ \ldots\ldots,\ j^{\mu_1-1},$$

and in this order the product of any two units ij^a and ij^b is linearly expressible in terms of the units which follow* the unit ij^{a+b-1}.

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* Shaw, *Trans. Am. Math. Soc.*, Vol. IV (1903), pp. 406–410.

It has been shown † that any number is expressible linearly in terms of elementary matrices, λ_{rst}, called "associative units," † where

$$\lambda_{rst}\lambda_{r's't'}=c\delta_{sr'}\lambda_{rs'(t+t')}, \qquad \mu_r-\mu_s\leq t<\mu_r, \quad \mu_{r'}-\mu_{s'}\leq t'<\mu_{r'},$$
$$\mu_r-1\geq t+t'\geq \mu_r-\mu_{s'}, \qquad \delta_{sr'}=1 \text{ if } s=r', \quad 0 \text{ if } s\neq r',$$

$c=1$ if $t+t'$ satisfies the above condition, 0 if $t+t'$ does not satisfy the condition, μ_r and μ_s are multiplicity numbers for r and s respectively, determined by the power of j which equals 0, and the power of j in ij^w which causes it to vanish, that is here $j^{\mu_1}=0$, $j^{\mu_1-1}\neq 0$ and $ij^{\mu_2}=0$, $ij^{\mu_2-1}\neq 0$.

The frame of the nilpotent system formed upon the base i and the adjunct unit j is:*

$$i=\lambda_{210}+a_{221}\lambda_{221}+a_{222}\lambda_{222}+\ldots\ldots+a_{22\mu_2-1}\lambda_{22\mu_2-1}+a_{12\mu_1-\mu_2}\lambda_{12\mu_1-\mu_2}+\ldots\ldots+a_{12\mu_1-1}\lambda_{12\mu_1-1},$$
$$j=\lambda_{111}+b_{221}\lambda_{221}+\ldots\ldots+b_{22\mu_2-1}\lambda_{22\mu_2-1}+b_{12\mu_1-\mu_2}\lambda_{12\mu_1-\mu_2}+\ldots\ldots+b_{12\mu_1-1}\lambda_{12\mu_1-1}.$$

We may indicate these expressions more compactly by introducing certain symbolic polynomials which will take care of the third subscripts. Thus we write

$$\lambda_{22}\theta A\equiv a_{221}\lambda_{221}+a_{222}\lambda_{222}+\ldots\ldots+a_{22\mu_2-1}\lambda_{22\mu_2-1},$$

where $A=a_{221}+a_{222}\theta+\ldots\ldots+a_{22\mu_2-1}\theta^{\mu_2-2}$, and θ is a nilpotent such that $\theta^{\mu_2}=0$, $\lambda_{22i}\theta=\lambda_{22i+1}$. In a similar manner we may write

$$\lambda_{12}\delta^{\mu_1-\mu_2}B=a_{12\mu_1-\mu_2}\lambda_{12\mu_1-\mu_2}+\ldots\ldots+a_{12\mu_1-1}\lambda_{12\mu_1-1},$$

where $B=a_{12\mu_1-\mu_2}+a_{12\mu_1-\mu_2+1}\delta+\ldots\ldots+a_{12\mu_1-1}\delta^{\mu_1-1}$, and δ is a nilpotent such that $\delta^{\mu_1}=0$, $\lambda_{12i}\delta=\lambda_{12i+1}$. We need to notice then that we may also indicate in the same way a series of terms in λ_{11i},

$$\lambda_{11}\delta G=a_{111}\lambda_{111}+a_{112}\lambda_{112}+\ldots\ldots+a_{11\mu_1-1}\lambda_{11\mu_1-1},$$

where $$G=a_{111}+a_{112}\delta+\ldots\ldots+a_{11\mu_1-1}\delta^{\mu_1-2},$$

and $$\lambda_{21}H=a_{210}\lambda_{210}+a_{211}\lambda_{211}+\ldots\ldots+a_{21\mu_2-1}\lambda_{21\mu_2-1},$$

where $$H=a_{210}+a_{211}\theta+\ldots\ldots+a_{21\mu_2-1}\theta^{\mu_2-1}.$$

Hence in this notation, if we omit λ and write only subscripts,

$$i=210+22\theta A+12\delta^{\mu_1-\mu_2}B, \quad j=111+22\theta C+12\delta^{\mu_1-\mu_2}E,$$

where

$$A(\theta)=a_0+a_1\theta+a_2\theta^2+\ldots\ldots+a_{\mu_2-2}\theta^{\mu_2-2}, \quad B(\delta)=b_0+b_1\delta+b_2\delta^2+\ldots\ldots+b_{\mu_2-1}\delta^{\mu_2-1},$$
$$C(\theta)=c_0+c_1\theta+c_2\theta^2+\ldots\ldots+c_{\mu_2-2}\theta^{\mu_2-2}, \quad E(\delta)=e_0+e_1\delta+e_2\delta^2+\ldots\ldots+e_{\mu_2-1}\delta^{\mu_2-1}.$$

If $\mu_1=\mu_2$, $b_0=b_1=0$ for 121 can not appear in i. We also use the same symbols for polynomials in j, the context always showing the use.

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* Shaw, *Trans. Am. Math. Soc.*, Vol. IV (1903), pp. 251–287.

The general form of the product ji is necessarily

$$ji = ijC(j) + E(j) \text{ in which } e_0 = e_1 = 0.$$

THEOREM I. *If* $C \neq 1$ *by a proper choice of* i *the product* ji *becomes* $ji = ijC(j)$.

PROOF: Let $X = x_1 j + x_2 j^2 + \ldots .$, then

$$j(i - X) = ijC + E - jX = (i - X)jC + E + jX(C - 1).$$

If we can determine X so that $E + jX(C-1) = 0$, we may set $i' = i - X$, whence $ji' = i'jC$, which is the form desired. We must have then

$$X = \frac{E}{j(1-C)} = \left(\frac{e_2 j + e_3 j^2 + \ldots . + e_{\mu_1 - 1} j^{\mu_1 - 2}}{1-C}\right).$$

By Lemma I, X is determinable provided the numerator does not start with a lower power of j than the denominator. This is evidently the case whenever $1-C$ starts with a term free from j, but when $C = 1 + j^x C_1$, where C_1 starts with a term free from j, then the numerator must start with a higher term than j^{x-1}. To show that in this case the desired situation always exists we proceed as follows:

Let us abbreviate the writing by setting temporarily

$$i^2 = ijA + B, \quad ji = ijC + j^w E_1,$$

where E_1 starts with a term free from j. Then we can easily see that if $P(j)$ is any polynomial in j,

$$P(j)i = iP(jC) + j^{w-1} E_1 \frac{P(jC) - P(j)}{C-1} \text{ when } C \neq 1,$$

where the division is purely formal and always possible.

We have now $j \cdot i^2 = jijA + jB = ij^2 AC +$ terms in j. But we have by changing the association,

$$\begin{aligned} ji \cdot i = ijCi + j^w E_1 i &= i\left[ijC \cdot C(jC) + j^{w-1} E_1 \frac{jC \cdot C(jC) - jC}{C-1}\right] + ij^w C^w E_1(jC) \\ &\qquad\qquad\qquad\qquad\qquad\qquad + \text{terms in } j \\ &= i^2 jC \cdot C(jC) + ij^w CE_1 \frac{C(jC) - 1}{C-1} + ij^w C^w E_1(jC) + \ldots . \\ &= ij^2 AC \cdot C(jC) + ij^w C\left[E_1 \frac{C(jC)-1}{C-1} + C^{w-1} E_1(jC)\right] + \ldots . . \end{aligned}$$

Since the terms in ij^2 must be equal, we have

$$ij^2\left[AC(-1 + C(jC)) + j^{w-2} C\left(E_1 \frac{C(jC)-1}{C-1} + C^{w-1} E_1(jC)\right)\right] = 0.$$

Now suppose that $C=1+j^xC_1$ where C_1 starts with a term free from j, then we have, since we may drop the factors that are invertible, by Lemma I.

$$ij^2\left[Aj^xC_1(jC)C^x+j^{w-2}\left(E_1\frac{C_1(jC)}{C_1}\cdot C^x+C^{w-1}E_1(jC)\right)\right]=0.$$

Now let $A=j^yA_1$ where A_1 starts with a term free from j, then the lowest terms from the two expressions that are multiplied by i are

$$j^{x+y+2}a_{10}c_{10}+j^w 2e_{10},$$

where a_{10} is the first term of A_1, c_{10} of C_1, and e_{10} of E_1, none of which are zero. Either these two cancel each other, which can happen only if

$$x+y+2=w, \text{ and hence } x\leq w-2;$$

or else each is not lower than μ_2. But we know that $x<\mu_2-1$, and hence if $\mu_2\leq w$ we have

$$x<\mu_2-1\leq w-1.$$

In any case then we must have $x\leq w-2$. Therefore whenever $C\neq 1$, $X(j)$ is determinable so that

$$ji=ijC(j).$$

Moreover, we have $X=j^{\mu_1-\mu_2}(X')$, and if $i'=i-X$, $i'j^{\mu_2}=ij^{\mu_2}-Xj^{\mu_2}=0$. Hence i' is of the same character as i, and the transformation is permissible. In other words, whenever $C\neq 1$, we may take $E=0$ and write,

$$i=210+22\theta A+12\delta^{\mu_1-\mu_2}B, \quad j=111+22\theta C.$$

We proceed to investigate first the case $C=1$.

IV. *Case I.* $C=1$.

If $C=1$ the generators have the form

$$i=210+22\theta A+12\delta^{\mu_1-\mu_2}B, \quad j=111+221+12\delta^{\mu_1-\mu_2}E.$$

Since, however, in this case

$$i^2=ijA+j^{\mu_1-\mu_2}B,$$

we may set $i'=i-\frac{1}{2}jA$, whence

$$ji'=ji+j^{\mu_1-\mu_2}E-\tfrac{1}{2}j^2A=i'j+j^{\mu_1-\mu_2}E,$$
$$i'^2=i^2-[ijA+\text{terms in powers of } j]=\text{terms in powers of } j.$$

Consequently, we may treat the case as if $A=0$, until we derive the types. We then return to the equations in i, j. The generators now become

$$i=210+12\delta^{\mu_1-\mu_2}B, \quad j=111+221+12\delta^{\mu_1-\mu_2}E.$$

From these we form the following products:

$$ij=211+22\theta^{\mu_1-\mu_2}E+12\delta^{\mu_1-\mu_2+1}B,\quad ij^n=21n+22\theta^{\mu_1-\mu_2+n-1}nE+12\delta^{\mu_1-\mu_2+n}B,$$

$$ijG(j)=21\theta G(\theta)+22\theta^{\mu_1-\mu_2}E[g_0+2g_1\theta+3g_2\theta^2+\ldots]+12\delta^{\mu_1-\mu_2}BG(\delta),$$

$$j^2=112+222+12\delta^{\mu_1-\mu_2+1}2E,\quad j^n=11\delta^n+22\theta^n+12\delta^{\mu_1-\mu_2+n-1}nE,$$

$$ji=211+12\delta^{\mu_1-\mu_2+1}B+11\delta^{\mu_1-\mu_2}E,$$

$$j^{\mu_1-\mu_2}=11\delta^{\mu_1-\mu_2}+22\theta^{\mu_1-\mu_2}+12\delta^{2\mu_1-2\mu_2-1}(\mu_1-\mu_2)E,$$

$$j^{\mu_1-\mu_2}E(j)=11\delta^{\mu_1-\mu_2}E(\delta)+22\theta^{\mu_1-\mu_2}E(\delta)+12\delta^{2\mu_1-2\mu_2-1}E[(\mu_1-\mu_2)e_0+\ldots],$$

$$\therefore\ ji=ij+j^{\mu_1-\mu_2}E(j)-22\theta^{\mu_1-\mu_2}E$$
$$-12\delta^{2\mu_1-2\mu_2-1}E[(\mu_1-\mu_2)e_0+(\mu_1-\mu_2+1)e_1\delta+\ldots],$$

The last two terms must vanish, so we have

$$\theta^{\mu_1-\mu_2}E=0, \tag{1}$$

$$\delta^{2\mu_1-2\mu_2-1}E[(\mu_1-\mu_2)e_0+\ldots]=0, \tag{2}$$

$$i^2=22\theta^{\mu_1-\mu_2}B+11\delta^{\mu_1-\mu_2}B$$
$$=j^{\mu_1-\mu_2}B(j)+22\theta^{\mu_1-\mu_2}B-22\theta^{\mu_1-\mu_2}B-12\delta^{2\mu_1-2\mu_2-1}E[(\mu_1-\mu_2)b_0+\ldots].$$

This gives rise to the equation

$$\delta^{2\mu_1-2\mu_2-1}E[(\mu_1-\mu_2)b_0+(\mu_1-\mu_2+1)b_1\theta+\ldots]=0. \tag{3}$$

The solutions of these three equations are in three classes:

1. $\mu_1-\mu_2\geq\mu_2$, *i. e.*, $\mu_1\geq 2\mu_2$, $\mu_1-\mu_2=\mu_2+\alpha$, $E\neq 0$.
2. $\mu_1-\mu_2<\mu_2$, $E\neq 0$.
3. $E=0$.

For Class 1 equation (1) is satisfied. Equation (2) becomes

$$\delta^{2\mu_2+2\alpha-1}E[(\mu_2+\alpha)e_0+\ldots]=0,\ \text{or}\ \delta^{\mu_1+\alpha-1}E[(\mu_2+\alpha)e_0+\ldots]=0.$$

This equation is satisfied unless $\alpha=0$ and then $E=\delta E_1$. Equation (3) becomes $\delta^{\mu_1+\alpha-1}E[(\mu_2+\alpha)b_0+\ldots]=0$, which is satisfied whenever (2) is.

For Class 2 let $E=\delta^v E_2$, where $e_{20}\neq 0$. Then (1) becomes $\theta^{\mu_1-\mu_2+v}E_2=0$.

$$\therefore\ \mu_1-\mu_2+v\geq\mu_2,\quad \mu_1+v\geq 2\mu_2.$$

Equation (2) becomes $\delta^{2\mu_1-2\mu_2+v-1}E_2[(\mu_1-\mu_2+v)e_{2v}.\ldots]=0$, which is satisfied unless $v=2\mu_2-\mu_1$, in which case $E_2=\delta E_3$ and $E=\delta^{v+1}E_3$, therefore $v>2\mu_2-\mu_1$. Equation (3) is satisfied with (2).

For Class 3 all three equations are satisfied. We have then for this case the following types:

11. $\mu_1>2\mu_2$, $\mu_1-\mu_2=\mu_2+\alpha$, $\alpha>0$, $E\neq 0$.
12. $\mu_1=2\mu_2$, $E=\delta E_1$, $E_1\neq 0$.
2. $\mu_1-\mu_2<\mu_2$, $E=\delta^v E_3$, $E_3\neq 0$, $v>2\mu_2-\mu$.
3. $E=0$.

These types are now written out as follows:

11. $\mu_1 > 2\mu_2$, $\mu_1 - \mu_2 = \mu_2 + \alpha$, $\alpha > 0$, $E \neq 0$. The generators for this type are

$$i = 210 + 12\delta^{\mu_2+\alpha}B, \quad j = 111 + 221 + 12\delta^{\mu_2+\alpha}E.$$

From these we get the following products:

$$ij = 211 + 12\delta^{\mu_2+\alpha+1}B, \quad j^2 = 112 + 222 + 12\delta^{\mu_2+\alpha+1}2E,$$
$$j^n = 11n + 22n + 12\delta^{\mu_2+\alpha+n-1}nE,$$
$$ji = 211 + 12\delta^{\mu_2+\alpha+1}B + 11\delta^{\mu_2+\alpha}E = ij + j^{\mu_2+\alpha}E(j), \quad i^2 = 11\delta^{\mu_2+\alpha}B = j^{\mu_2+\alpha}B(j).$$

12. $\mu_1 = 2\mu_2$, $E = \delta E_1$, $E_1 \neq 0$:

$$i = 210 + 12\delta^{\mu_1}B, \quad j = 111 + 221 + 12\delta^{\mu_2+1}E_1, \quad ij = 211 + 12\delta^{\mu_2+1}B,$$
$$j^n = 11n + 22n + 12\delta^{\mu_2+n}nE_1,$$
$$ji = 211 + 12\delta^{\mu_2+1}B + 11\delta^{\mu_2+1}E_1 = ij + j^{\mu_2+1}E_1(j), \quad i^2 = 11\delta^{\mu_2}B = j^{\mu_2}B(j).$$

2. $\mu_1 - \mu_2 < \mu_2$, $E = \delta^v E_3$, $E_3 \neq 0$, $v > 2\mu_2 - \mu_1$:

$$i = 210 + 12\delta^{\mu_1-\mu_2}B, \quad j = 111 + 221 + 12\delta^{\mu_1-\mu_2+v}E_3,$$
$$ij = 211 + 12\delta^{\mu_1-\mu_2+1}B, \quad j^n = 11n + 22n + 12\delta^{\mu_1-\mu_2+v+n-1}nE_3,$$
$$ji = 211 + 12\delta^{\mu_1-\mu_2+1}B + 11\delta^{\mu_1-\mu_2+v}E_3 = ij + j^{\mu_1-\mu_2+v}E_3(j),$$
$$i^2 = 22\theta^{\mu_1-\mu_2}B + 11\delta^{\mu_1-\mu_2}B = j^{\mu_1-\mu_2}B(j).$$

3. $E = 0$:

$$i = 210 + 12\delta^{\mu_1-\mu_2}B, \quad j = 111 + 221, \quad ij = 211 + 12\delta^{\mu_1-\mu_2+1}B,$$
$$ji = 211 + 12\delta^{\mu_1-\mu_2+1}B = ij, \quad j^n = 11n + 22n,$$
$$i^2 = 11\delta^{\mu_1-\mu_2}B + 22\theta^{\mu_1-\mu_2}B = j^{\mu_1-\mu_2}B(j).$$

Type	$ji' =$	$(i')^2 =$
1	$i'j,$	$j^{\mu_1-\mu_2}B(j),$
2	$i'j + j^{\mu_2+1}E_1(j),$	$j^{\mu_2}B(j),$
3	$i'j + j^{\mu_2+\alpha}E(j),$	$j^{\mu_2+\alpha}B(j),$
4	$i'j + j^{\mu_1-\mu_2+v}E_3(j),$	$j^{\mu_1-\mu_2}B(j),$

or in terms of i and j, instead of i' and j.

Type	$ji =$	$i^2 =$
1	$ij,$	$ijA + \frac{1}{2}j^{\mu_1-\mu_2}E(j)(jA)' - \frac{1}{4}j^2A^2 + j^{\mu_1-\mu_2}B(j),$
2	$ij + j^{\mu_2+1}E_1(j),$	$ijA + \frac{1}{2}j^{\mu_1-\mu_2}E(j)(jA)' - \frac{1}{4}j^2A^2 + j^{\mu_2}B(j),$
3	$ij + j^{\mu_2+\alpha}E(j),$	$ijA + \frac{1}{2}j^{\mu_1-\mu_2}E(j)(jA)' - \frac{1}{4}j^2A^2 + j^{\mu_1-\mu_2}B(j),$
4	$ij + j^{\mu_1-\mu_2+v}E_3(j),$	$ijA + \frac{1}{2}j^{\mu_1-\mu_2}E(j)(jA)' - \frac{1}{4}j^2A^2 + j^{\mu_1-\mu_2}B(j),$

when $(jA)'$ is the formal derivative as to j of the polynomial jA, and A is arbitrary. In case A starts with a term of order $\mu_1 - \mu_2 - 1$ or higher, i' may be used for i.

V. *Case II.* $A=0,\ C\neq 1,\ E=0.$

If A vanishes we have the defining equations:

$$i=210+12\delta^{\mu_1-\mu_2}B,\quad j=111+22\theta C.$$

Then $ij^s=21s+12\delta^{\mu_1-\mu_2+s}BC^s$, and $iP(j)=21P(\theta)+12\delta^{\mu_1-\mu_2}BP(\delta C)$.

Again $ji=21\theta C+12\delta^{\mu_1-\mu_2+1}B=ijC(j)-12\delta^{\mu_1-\mu_2+1}BC\cdot C(\delta C)+12\delta^{\mu_1-\mu_2+1}B$

$$\therefore\ \delta^{\mu_1-\mu_2+1}B[1-C\cdot C(\delta C)]=0. \tag{1}$$

Also $j^2=112+22\theta^2C^2$, ..., whence $P(j)=11P+22P(\theta C)$,

$i^2=11\delta^{\mu_1-\mu_2}B+22\theta^{\mu_1-\mu_2}B$,

but $j^{\mu_1-\mu_2}=11\delta^{\mu_1-\mu_2}+22\theta^{\mu_1-\mu_2}C^{\mu_1-\mu_2}$, so $i^2=j^{\mu_1-\mu_2}B(j)$,

$$\therefore\ \theta^{\mu_1-\mu_2}[B-C^{\mu_1-\mu_2}B(\theta C)]=0. \tag{2}$$

Let C be written $C=c_0+\theta^tC_1$ where c_0 may be 0, but $c_{10}\neq 0$ and $0<t<\mu_2-1$. Substituting in equation (1)

$$\delta^{\mu_1-\mu_2+1}B[1-(c_0+\theta^tC_1)(c_0+\theta^tC^tC_1(\theta C))]=0,$$

or

$$\delta^{\mu_1-\mu_2+1}B[1-c_0^2-\theta^tc_0C_1-c_0\theta^tC^tC_1(\theta C)]=0.$$

By applying Lemma II we have either

11. $c_0^2\neq 1$ and $B=\delta^{\mu_2-1}b_{\mu_2-1}$,

or 12. $c_0^2=1$ and $B=\delta^{\mu_2-t-1}B_1$.

Now substituting in equation (2) the results just found, we have for 11:

$$\theta^{\mu_1-1}b_{\mu_2-1}[1-(c_0+\theta^tC_1)^{\mu_1-1}]=0.$$

Therefore by Lemma II,

111, $b_{\mu_2-1}=0$, and $B=0$; or 112, $c_0^{\mu_1-1}=1$, $b_{\mu_2-1}\neq 0$.

For the case 12, $\theta^{\mu_1-t-1}[B_1-C^{\mu_1-t-1}B_1(\theta C)]=0$.

Hence either 121, for $c_0=1$, $\theta^{\mu_1-1}b_{10}(\mu_1-t-1)C_1=0$, for which we have either

1211, $\mu_1>\mu_2$, $B=\theta^{\mu_2-t-1}B_1$,

or 1212, $\mu_1=\mu_2$, $B=\theta^{\mu_2-t}B_2$,

or 122, for $c_0=-1$, $\theta^{\mu_1-t-1}[B_1-(-1+\theta^tC_1)^{\mu_1-t-1}B_1(\theta C)]=0$,

for which we have

1221, μ_1-1 is even, $B_1=\theta^tB_3$, *i. e.*, $B=\theta^{\mu_2-1}b_{\mu_2-1}$, $b_{\mu_2-1}\neq 0$,

or 1222, $B_1=\theta^{t+1}B_4$, *i. e.*, $B=0$,

or 1223, $\mu_1-t-1\geq\mu_2$, $B_1\neq 0$.

Consequently when $A=0$ the following seven types arise:

$$A=0, \quad C=c_0+\theta^t C_1,$$

111. $c_0^2 \neq 1, \quad B=0.$
112. $c_0^{\mu_1-1}=1, \quad B=\delta^{\mu_2-1}b_{\mu_2-1}.$
1211. $c_0=1, \quad B=\delta^{\mu_2-t-1}B_1, \quad \mu_1>\mu_2.$
1212. $c_0=1, \quad B=\delta^{\mu_2-t}B_2, \quad \mu_1=\mu_2.$
1221. $c_0=-1, \quad \mu_1-1$ is even, $B=\delta^{\mu_2-1}b_{\mu_2-1}.$
1222. $c_0=-1, \quad B=0.$
1223. $c_0=-1, \quad \mu_1-t-1 \geq \mu_2, \quad B=\delta^{\mu_2-t-1}B_1, \quad B_1 \neq 0.$

We now work out these seven types obtaining the expressions for the products ji and i^2.

111. $C=c_0+\theta^t C_1, \quad c_0^2 \neq 1, \quad B=0.$ For this the generators have the form
$i=210, \quad j=111+22\theta C.$

Then $ij=211, \quad ji=21\theta C=ijC(j)$, and $i^2=0.$

112. $c_0^{\mu_1-1}=1, \quad B=\delta^{\mu_2-1}b_{\mu_2-1}.$

Now $i=210+12\delta^{\mu_1-1}b_{\mu_2-1}, \quad j=111+22\theta C, \quad ij=211, \quad ji=21\theta C=ijC(j),$
$i^2=11\delta^{\mu_1-1}b_{\mu_1-1}+22\theta^{\mu_1-1}b_{\mu_2-1}, \quad j^n=11\delta^n+22\theta^n C^n, \quad \therefore\ i^2=j^{\mu_1-1}b_{\mu_2-1}.$

1211. $c_0=1, \quad B=\delta^{\mu_2-t-1}B_1, \quad \mu_1>\mu_2.$
$i=210+12\delta^{\mu_1-t-1}B_1, \quad j=111+221+22\theta^{t+1}C_1, \quad ij^n=21n+12\delta^{\mu_1-t+n-1}B_1,$
$ji=211+12\delta^{\mu_1-t}B_1+21\theta^{t+1}C_1=ij+ij^{t+1}C_1(j),$
$j^n=11n+22\theta^n C^n=11n+22\theta^n(1+\theta^t C_1)^n,$
$i^2=11\delta^{\mu_1-t-1}B_1+22\theta^{\mu_1-t-1}B_1=j^{\mu_1-t-1}B_1(j).$

1212. $c_0=1, \quad B=\delta^{\mu_2-2}B_2, \quad \mu_1=\mu_2.$
$i=210+12\delta^{\mu_1-t}B_2, \quad j=111+221+22\theta^{t+1}C_1=111+22\theta C,$
$ij^n=21n+12\delta^{\mu_1-t+n}B_2, \quad ji=211+21\theta^{t+1}C_1+12\delta^{\mu_1-t+1}B_2=ij+ij^{t+1}C_1(j),$
$i^2=11\delta^{\mu_1-t}B_2+22\theta^{\mu_1-t}B_2=j^{\mu_1-t}B_2(j).$

1221. μ_1-1 is even, $B=\delta^{\mu_2-1}b, \quad c_0=-1.$
$i=210+12\delta^{\mu_1-1}b, \quad j=111+221(-1)+22\theta^{t+1}C_1, \quad ij=211,$
$ji=211(-1)+21\theta^{t+1}C_1=-ij+ij^{t+1}C_1(j),$
$j^n=11n+22\theta^n(-1+\theta^t C_1)^n, \quad i^2=11\delta^{\mu_1-1}b+22\theta^{\mu_1-1}b=j^{\mu_1-1}b.$

1222. $c_0=-1, \quad B=0.$
$i=210, \quad j=111+221(-1)+22\theta^{t+1}C_1, \quad ij=211,$
$ji=211(-1)+21\theta^{t+1}C_1=-ij+ij^{t+1}C_1(j), \quad i^2=0.$

1223. $c_0=-1,\quad \mu_1-t-1\geq\mu_2,\quad B=\delta^{\mu_1-t-1}B_1,\quad B_1\neq 0.$

$$i=210+12\delta^{\mu_1-t-1}B_1,\quad j=111+221(-1)+22\theta^{t+1}C_1,$$
$$ij^n=21n+12\delta^{\mu_1-t+n-1}(-1)^nB_1,$$
$$ji=211(-1)+21\theta^{t+1}C_1+12\delta^{\mu_1-t}B_1=-ij+ij^{t+1}C_1(j),$$
$$i^2=11\delta^{\mu_1-t-1}B_1=j^{\mu_1-t-1}B_1(j).$$

Type	$ji=$	$i^2=$	
1	$ij[c_0+j^tC_1(j)]$,	0,	$c_0^2\neq 1$,
2	$ij[c_0+j^tC_1(j)]$,	$j^{\mu_1-1}b_{\mu_2-1}$,	$c_0^2\neq 1,\quad b_{\mu_2-1}\neq 0$,
3	$ij+ij^{t+1}C_1(j)$,	$j^{\mu_1-t-1}B_1(j)$,	$\mu_1>\mu_2$,
4	$ij+ij^{t+1}C_1(j)$,	$j^{\mu_1-t}B_2(j)$,	$\mu_1=\mu_2$,
5	$-ij+ij^{t+1}C_1(j)$,	$j^{\mu_1-1}b$,	μ_1 is odd, $b\neq 0$,
6	$-ij+ij^{t+1}C_1(j)$,	0,	
7	$-ij+ij^{t+1}C_1(j)$,	$j^{\mu_1-t-1}B_1(j)$,	$\mu_1-t-1\geq\mu_2,\quad B_1\neq 0$.

VI. *Case III.* $A\neq 0,\quad C\neq 1,\quad E=0.$

In this case, which is the most general, we first form the following products from the generators:

$$i=210+22\theta A+12\delta^{\mu_1-\mu_2}B,\quad j=111+22\theta C,$$
$$ij^r=21r+22\theta^{r+1}AC^r+12\delta^{\mu_1-\mu_2+r}BC^r,\quad r<\mu_2-1,$$
$$ij^{\mu_2-1}=21\theta^{\mu_2-1}+12\delta^{\mu_1-1}BC^{\mu_2-1},$$

$iP(j)=210P(\theta)+22\theta AP(\theta C)+12\delta^{\mu_1-\mu_2}BP(\delta C)$, where $P(j)$ is a polynomial in j.

$$j^s=11s+22\theta^sC^s,\quad s<\mu_2,\quad j^s=11\theta^s,\quad \mu_1>s\geq\mu_2,$$
$$P(j)=11P(\delta)+22P(\theta C),$$
$$\begin{aligned}ji&=21\theta C+22\theta^2AC+12\delta^{\mu_1-\mu_2+1}B\\&=ijC(j)+22\theta^2AC+12\delta^{\mu_1-\mu_2+1}B-22\theta^2AC\cdot C(\theta C)-12\delta^{\mu_1-\mu_2+1}BCC(\theta C),\end{aligned}$$
$$P(j)\cdot i\theta=210P(\theta C)+22\theta AP(\theta C)+12\delta^{\mu_1-\mu_2}BP(\delta)=i\cdot P(jC),$$
$$\begin{aligned}i^2&=21\theta A+22\theta^2A^2+22\theta^{\mu_1-\mu_2}B+11\delta^{\mu_1-\mu_2}B+12\delta^{\mu_1-\mu_1+1}AB\\&=ijA(j)+j^{\mu_1-\mu_2}B(j)+22\theta^2A^2+22\theta^{\mu_1-\mu_2}B+12\delta^{\mu_1-\mu_2+1}AB\\&\qquad-22\theta^2ACA(\theta C)-22\theta^{\mu_1-\mu_2}C^{\mu_1-\mu_2}B(\theta C)-12\delta^{\mu_1-\mu_2}BCA(\theta C).\end{aligned}$$

Since the terms in 22( ) and 12( ) must vanish in both the products ji and i^2, we have four equations:

$$\left.\begin{aligned}&(1)\quad \theta^2AC[1-C(\theta C)]=0,\\&(2)\quad \delta^{\mu_1-\mu_2+1}B[1-C\cdot C(\delta C)]=0,\\&(3)\quad \delta^{\mu_1-\mu_2+1}B[A-C\cdot A(\delta C)]=0,\\&(4)\quad \theta^2A[A-C\cdot A(\theta C)]+\theta^{\mu_1-\mu_2}[B-C^{\mu_1-\mu_2}B(\theta C)]=0.\end{aligned}\right\}\quad\text{(A)}$$

Before reducing these we need a second theorem which we proceed to deduce.

If $A(j)=j^sA_1(j)$, $0\leq s<\mu_1-1$, then $i^2=ij^{s+1}A_1(j)+j^{\mu_1-\mu_2}B(j)$, where $a_{10}\neq 0$. By Lemma III we may now express a polynomial in j in terms of one in jC if $C=c_0+jC_1$. Let $A_1(j)=A_1'(jC)$ and determine i' from $i=i'A_1'(j)$ where $a_{10}\neq 0$, then

$$i^2=i'A_1'i'A_1'=i'j^{s+1}A_1(j)\cdot A_1'(j)+j^{\mu_1-\mu_2}B(j).$$

But using the formula $P(j)i=iP(jC)$,

$$A_1'(j)\cdot i=iA_1'(jC)=iA_1(j),$$
$$A_1'(j)\cdot i'=iA_1(j)A_1'^{-1}(j)=i'A_1(j).$$

Thence $\quad i'A_1'i'A_1'=i'^2A_1A_1'=i'j^{s+1}A_1(j)A_1'(j)+j^{\mu_1-\mu_2}B(j).$

That is $\quad i'^2=i'j^{s+1}+j^{\mu_1-\mu_2}B(j)A_1^{-1}(j)A_1'^{-1}(j)=i'j^{s+1}+j^{\mu_1-\mu_2}B'.$

THEOREM II. *If $C(j)=c_0+jC_1(j)$, where $c_0\neq 0$ then $j=P(jC)$ and $A_1'(j)$ is known, and invertible. Hence we may choose i so that the term $ij^{s+1}A_1(j)$ becomes simply ij^{s+1}, that is, we may take $A_1=1$.*

As a consequence of Theorem II we have two sub-cases,

III_1. $A(j)=j^s$, $C(j)=c_0+jC_1(j)$, $c_0\neq 0$.
III_2. $C(j)=j^tC_1(j)$, $t>0$, $c_{10}\neq 0$.

Case III_1. The generators may now be written

$$i=210+22\theta^{s+1}+12\delta^{\mu_1-\mu_2}B,\quad j=111+22\theta C,$$

and we shall have

$$ji=ijC(j),\quad i^2=ij^{s+1}+j^{\mu_1-\mu_2}B(j),$$

where the following four equations, which are the reduced forms of equations (A), must be simultaneously satisfied:

$$\left.\begin{aligned}
&(1)'\quad \theta^{s+2}C[1-C(\theta C)]=0,\\
&(2)'\quad \delta^{\mu_1-\mu_2+1}B[1-C\cdot C(\delta C)]=0,\\
&(3)'\quad \delta^{\mu_1-\mu_2+s+1}[1-C^{s+1}]=0,\\
&(4)'\quad \theta^{2s+2}[1-C^{s+1}]+\theta^{\mu_1-\mu_2}[B-C^{\mu_1-\mu_2}B(\theta C)]=0.
\end{aligned}\right\}\qquad (A)'$$

Consider equation (1)',

$$\theta^{s+2}C[1-C(\theta C)]=0.$$

This leads to two classes 3 and 4.

3 $\quad s+2=\mu_2$, $C=c_0+\theta^tC_1$, $c_0\neq 0$, $c_{10}\neq 0$.

4 $\quad s+2<\mu_2$, then since $c_0\neq 0$, $1-C(\theta C)=-\theta^{\mu_2-s-2+r}C_2$, or $C(\theta C)=1+\theta^{\mu_2-s-2+r}C_2$, $c_{20}\neq 0$. $\therefore$ $c_0+\theta^tC_1(\theta C)C^t=1+\theta^{\mu_2-s-2+r}C_2$,

or $\quad c_0+\theta^t[c_{10}+\theta C\cdot C_3(\theta C)][c_0^t+\theta^tC_4]=1+\theta^{\mu_2-s-2+r}C_2,$

or $\quad c_0+\theta^tc_{10}c_0^t+\theta^{t+1}C_5=1+\theta^{\mu_2-s-2+r}C_2.$

Therefore $c_0=1$ and $t=\mu_2-s-2+r$ since $c_0^t\neq 0\neq c_{10}$ and $C(\theta C)=1+\theta^t C_2$, $t<\mu_2$. Consider next equation (3)′,

$$\delta^{\mu_1-\mu_2+s+1}B[1-C^{s+1}]=0.$$

For class 3, $s+2=\mu_2$, $C=c_0+\theta^t C_1$, $c_0\neq 0$, $c_{10}\neq 0$ this becomes

$$\delta^{\mu_1-1}b_0[1-C^{\mu_2-1}]=0,$$

from which we have either

31 $$b_0=0,\quad B=\delta B_1,\quad c_0^{\mu_2-1}\neq 1,$$

or 32 $$c_0^{\mu_2-1}=1.$$

For class 4, $s+2<\mu_2$, $c_0=1$, $t=\mu_2-s-2+r$ equation (3)′ becomes

$$\delta^{\mu_1-\mu_2+s+1}B[1-(1+\delta^{\mu_2-s-2+r}C_1)^{s+1}]=0,$$

$$\delta^{\mu_1-1+r}B[-C_1(s+1)+\ldots.]=0,\quad \delta^{\mu_1-1+r}B=0.$$

This is satisfied unless $r=0$, then $B=\delta B_1$. Consider now equation (2)′,

$$\delta^{\mu_1-\mu_2+1}B[1-C\cdot C(\delta C)]=0.$$

For 31 $s+2=\mu_2$, $c_0^{\mu_2-1}\neq 1$, $C=c_0+\delta^t C_1$, $c_0\neq 0\neq c_{10}$. Put $B=\delta^v B_2$, $b_{20}\neq 0$. Then $\delta^{\mu_1-\mu_2+v+1}B_2[1-C\cdot C(\delta C)]=0=\delta^{\mu_1-\mu_2+v+1}[1-C\cdot C(\delta C)]$.

311 If $c_0^2\neq 1$ then $v\geq\mu_2-1$, $B=\delta^{\mu_2-1}b_{\mu_2-1}$, or $B=0$ if $b_{\mu_2-1}=0$.

312 If $c_0^2=1$, $c_0^{\mu_2-1}\neq 1$, that is $c_0=-1$ and μ_2-1 is odd, then

$$\mu_1-\mu_2+v+t+1\geq\mu_1.\quad\therefore\; v\geq\mu_2-t-1.$$

For 32 $c_0^{\mu_2-1}=1$, then $\delta^{\mu_1-\mu_2+1}B[1-C\cdot C(\delta C)]=0$,

$$\delta^{\mu_1-\mu_2+1}B[1-(c_0+\delta^t C_1)(c_0+\delta^t C^t C_1(\delta C)]=0.$$

321 If $c_0^2\neq 1$, $B=\delta^{\mu_2-1}b_{\mu_2-1}$.

322 If $c_0^2=1$, $B=\delta^{\mu_2-t-1}B_3$, and if $c_0=-1$, μ_2-1 is even, that is μ_2 is odd.

For Class 4, $s+2<\mu_2$, $c_0=1$, $t=\mu_2-s-2+r$, equation (2)′ becomes

$$\delta^{\mu_1-\mu_2+1}B[1-C\cdot C(\delta C)]=0,\quad \delta^{\mu_1-\mu_2+1}B[1-(1+\delta^t C_1)(1+\delta^t C^t C_1(\delta C)]=0,$$

$$\therefore\; \delta^{\mu_1-\mu_2+1+t}B=0, \text{ which is } \delta^{\mu_1-s-1+r}B=0.\quad\therefore\; B=\delta^{s+1-r}B_4.$$

We consider finally equation (4)′,

$$\theta^{2s+2}[1-C^{s+1}]+\theta^{\mu_1-\mu_2}[B-C^{\mu_1-\mu_2}B(\theta C)]=0.$$

For 311 $c_0^2\neq 1$, $B=\delta^{\mu_2-1}b$ where b may be 0, $s+2=\mu_2$, $c_0^{\mu_2-1}\neq 1$, $c_0\neq 0\neq c_{10}$, then $\theta^{\mu_1-1}b[1-(c_0+\theta^t C_1)^{\mu_1-1}]=0$, which is satisfied unless $\mu_1=\mu_2$ since $\theta^{\mu_2}=0$. Therefore we have also

3111 $\mu_1=\mu_2$, $b=b^*_{\mu_2-1}=0$, *i. e.*, $v>\mu_2-1$, and

3112 $\mu_1=\mu_2$, $c_0^{\mu_1-1}=1$, but this is impossible since it is a contradiction to 311.

* We write b in place of b_{μ_2-1} in what follows.

For 312 $c_0=-1$, μ_2-1 is odd, $B=\theta^v B_2$, $b_{20}\neq 0$, $v>\mu_2-t-1$, equation (4)′ becomes

$$\theta^{\mu_1-\mu_2+v}[B_2-(-1+\theta C_1)^{\mu_1-\mu_2+v}B_2(\theta C)]=0.$$

We have then either

3121, $\mu_1-\mu_2+v$ is odd, and $\mu_1-\mu_2+v\geq\mu_2$, *i. e.*, $v\geq 2\mu_2-\mu_1$, or
3122, $\mu_1-\mu_2+v$ is even.

Now $$\theta^{\mu_1-\mu_2+v}[B_2-B_2(\theta C)-\ldots\ldots]=0$$

becomes $$\theta^{\mu_1-\mu_2+v+1}[b_{21}(1-C)+\ldots\ldots]=0,$$

and $$\mu_1-\mu_2+v+1\geq\mu_2,\ i.\,e.,\ v\geq 2\mu_2-\mu_1-1.$$

For 321 $c_0^2\neq 1$, $B=\delta^{\mu_2-1}b_{\mu_2-1}$, $c_0^{\mu_2-1}=1$, equation (4)′ becomes

$$\theta^{\mu_1-1}b_{\mu_2-1}[1-(c_0+\theta^t C_1)^{\mu_1-1}]=0$$

which is satisfied since $\theta^{\mu_2}=0$. If $\mu_1>\mu_2$, $\theta^{\mu_1-1}=0$. If $\mu_1=\mu_2$, the [] causes the expression to vanish.

For 322 $c_0^2=1$, $B=\delta^{\mu_2-t-1}B_3$, μ_2-1 is even if $c_0=-1$ and (4)′ becomes $\theta^{\mu_1-t-1}[B_3-(c_0+\theta^t C_1)^{\mu_1-t-1}B_3(\theta C)]=0$, for which we have

3221, $c_0=1$, $\theta^{\mu_1-t-1}[b_{31}\theta(1-C)+\ldots\ldots-\theta^t C_1 B_3(\theta C)-\ldots\ldots]=0$,
$\therefore\ \theta^{\mu_1-1}B_3(\theta C)=0$, which is satisfied unless $\mu_1=\mu_2$. Then
32211, $\mu_1>\mu_2$.
32212, $\mu_1=\mu_2$, $B_3(\theta C)=\theta B_5$, *i. e.*, $B=\theta^{\mu_2-t}B$.
3222, $c_0=-1$, μ_2-1 is even, then

$$\theta^{\mu_1-t-1}[B_3-(-1)^{\mu_1-t-1}B_3(\theta C)-(-1)^{\mu_1-t}\theta^t C_1 B_3(\theta C)+\ldots\ldots]=0.$$

32221. If μ_1-t-1 is even, $\theta^{\mu_1-t-1}[b_{31}\theta(1-C)+\ldots\ldots]=0$.
$\therefore\ B_3=\theta^t B_6$, which makes $B=\theta^{\mu_2-1}b_{\mu_2-1}$.
32222. If μ_1-t-1 is odd, $B_3=\theta^{t+1}B_7$, which makes $B=0$.

For Class 4 $s+2<\mu_2$, $c_0=1$, $t=\mu_2-s-2+r$, $B=\theta^{s+1-r}B_4$.
Equation (4)′ becomes

$$\theta^{2s+2}[1-C^{s+1}]+\theta^{\mu_1-\mu_2}[B-C^{\mu_1-\mu_2}B(\theta C)]=0,$$
$$\theta^{\mu_1-\mu_2+s+1-r}[B_4-(1+\theta^t C_1)^{\mu_1-\mu_2+s+1-r}B_4(\theta C)]=0,$$
$$\theta^{\mu_1-\mu_2+s+1-r}[b_{41}\theta(1-C)+\ldots\ldots+\theta^t C_1 B_4(\theta C)+\ldots\ldots]=0,$$
$$\theta^{\mu_1-1}[b_{41}\theta+b_{42}\theta^2(1+C)+\ldots\ldots+C_1 B_4(\theta C)+\ldots\ldots]=0,$$
$\therefore\ \theta^{\mu_1-1}B_4(\theta C)=0$, for which we have either

41, $\mu_1>\mu_2$, or
42, $\mu_1=\mu_2$, $B_4=\theta B_6$, *i. e.*, $B=\theta^{s+2-r}B_6$.

We have therefore in Case III_1 the following eleven types:

$$s+2=\mu_2\left\{\begin{array}{rl}
311. & \mu_1>\mu_2,\quad B=\theta^{\mu_2-1}b_{\mu_2-1},\quad c_0^{\mu_2-1}\neq 1,\quad c_0^2\neq 1.\\
3111. & \mu_1=\mu_2,\quad B=0,\quad c_0^2\neq 1,\quad c_0^{\mu_2-1}\neq 1.\\
3121. & c_0=-1,\ c_0^{\mu_2-1}\neq 1,\ \mu_2-1 \text{ is odd},\ B=\theta^v B_2,\ \mu_1-\mu_2+v \text{ is odd},\\
 & \qquad v\geq 2\mu_2-\mu_1,\quad v>\mu_2-t-1,\quad \mu_1-\mu_2\geq t+1.\\
3122. & c_0=-1,\ c_0^{\mu_2-1}\neq 1,\ \mu_2-1 \text{ is odd},\ B=\theta^v B_2,\ \mu_1-\mu_2+v \text{ is even},\\
 & \qquad v\geq 2\mu_2-\mu_1-1.\\
321. & c_0^2\neq 1,\quad c_0^{\mu_2-1}=1,\quad B=\theta^{\mu_2-1}b_{\mu_2-1}.\\
32211. & \mu_1>\mu_2,\quad c_0=1,\quad B=\theta^{\mu_2-t-1}B_3.\\
32212. & \mu_1=\mu_2,\quad c_0=1,\quad B=\theta^{\mu_2-t}B_5.\\
32221. & c_0=-1,\ \mu_2-1 \text{ is even},\ \mu_1-t-1 \text{ is even},\ B=\theta^{\mu_2-1}b_{\mu_2-1}.\\
32222. & c_0=-1,\ \mu_2-1 \text{ is even},\ \mu_1-t-1 \text{ is odd},\ B=0.
\end{array}\right.$$

$$s+2>\mu_2\left\{\begin{array}{rl}
41. & \mu_1>\mu_2,\quad c_0=1,\quad t=\mu_2-s-2+r,\quad B=\theta^{s+1-r}B_4.\\
42. & \mu_1=\mu_2,\quad c_0=1,\quad t=\mu_2-s-2+r,\quad B=\theta^{s+2-r}B_6.
\end{array}\right.$$

These particular types are written out as follows:

311. $s+2=\mu_2,\quad \mu_1>\mu_2,\quad c_0^2\neq 1,\quad B=\theta^{\mu_2-1}b_{\mu_2-1},\quad C=c_0+\theta^t C_1.$

$$i=210+22\theta^{\mu_2-1}+12\delta^{\mu_1-1}b,\quad j=111+22\theta C.$$

Then $ij=211,\quad ji=21\theta C=ijC(j)=c_0 ij+ij^{t+1}C_1(j),$

$$i^2=21\theta^{\mu_2-1}+11\delta^{\mu_1-1}b=ij^{\mu_2-1}+j^{\mu_1-1}b.$$

3111. $s+2=\mu_2,\quad \mu_1=\mu_2,\quad B=0,\quad c_0^2\neq 1,\quad c^{\mu_2-1}\neq 1.$

$$i=210+22\theta^{\mu_2-1},\quad j=111+22\theta C.$$

Then $ij=211,\quad ji=21\theta C=ijC(j)=c_0 ij+ij^{t+1}C_1(j),\quad i^2=21\theta^{\mu_2-1}=ij^{\mu_2-1}.$

3121. $c_0=-1,\ c_0^{\mu_2-1}\neq 1,\ \mu_2-1$ is odd, $B=\theta^v B_2,\ \mu_1-\mu_2+v$ is odd, $v\geq 2\mu_2-\mu_1,$ $v>\mu_2-t-1,\ \mu_1-\mu_2\geq t+1.$

$$\begin{aligned}
i&=210+22\theta^{\mu_2-1}+12\delta^{\mu_1-\mu_2+v}B_2,\\
j&=111+22(-1)+22\theta^{t+1}C_1=111+22\theta C,\quad ij^n=21n+12\delta^{\mu_1-\mu_2+v+n}(-1)^n B_2,\\
ji&=211(-1)+12\delta^{\mu_1-\mu_2+v+1}B_2+21\theta^{t+1}C_1=-ij+ij^{t+1}C_1(j),\\
j^n&=11n+22\theta^n C^n=11n+22\theta^n(-1+\theta^t C_1)^n,\quad j^{\mu_1-\mu_2+v}=11\delta^{\mu_1-\mu_2+v},\\
i^2&=21\theta^{\mu_2-1}+22\theta^{\mu_1-\mu_2+v}B_2+12\delta^{\mu_1+v-1}B_2+11\delta^{\mu_1-\mu_2+v}B_2\\
&=ij^{\mu_2-1}+j^{\mu_1-\mu_2+v}B_2(j).
\end{aligned}$$

3122. $c_0=-1,\quad c_0^{\mu_2-1}\neq 1,\quad \mu_2-1$ is odd, $B=\theta^v B_2,\quad \mu_1-\mu_2+v$ is even, $v\geq 2\mu_2-\mu_1-1,\ v>\mu_2-t-1.$

$$\begin{aligned}
i&=210+22\theta^{\mu_2-1}+12\delta^{\mu_1-\mu_2+v}B_2,\\
j&=111+221(-1)+22\theta^{t+1}C_1=111+22\theta C,\quad ij^n=21n+12\delta^{\mu_1-\mu_2+v+n}(-1)^n B_2,\\
ji&=211(-1)+21\theta^{t+1}C_1+12\delta^{\mu_1-\mu_2+v+1}B_2=-ij+ij^{t+1}C_1(j),\\
j^n&=11n+22\theta^n(-1+\theta^t C_1)^n,\quad j^{\mu_1-\mu_2+v}=11\delta^{\mu_1-\mu_2+v},\\
i^2&=21\theta^{\mu_2-1}+11\delta^{\mu_1-\mu_2+v}B_2+12\delta^{\mu_1+v-1}B_2=ij^{\mu_2-1}+j^{\mu_1-\mu_2+v}B_2(j).
\end{aligned}$$

321. $c_0^2 \neq 1, \quad c_0^{\mu_2-1}=1, \quad B=\theta^{\mu_2-1}b, \quad s+2=\mu_2.$

$i=210+22\theta^{\mu_2-1}+12\delta^{\mu_1-1}b, \quad j=111+22\theta C, \quad ij=211,$

$ji=21\theta C=ijC(j), \quad i^2=21\theta^{\mu_2-1}+22\theta^{\mu_1-1}b+11\delta^{\mu_1-1}b=ij^{\mu_2-1}+j^{\mu_1-1}b.$

32211. $c_0=1, \quad \mu_1>\mu_2, \quad B=\theta^{\mu_2-t-1}B_3, \quad s+2=\mu_2.$

$i=210+22\theta^{\mu_2-1}+12\delta^{\mu_1-t-1}B_3,$

$j=111+221+22\theta^{t+1}C_1, \quad ij^n=21n+12\delta^{\mu_1-t+n-1}B_3,$

$ji=211+12\delta^{\mu_1-t}B_3+21\theta^{t+1}C_1=ij+ij^{t+1}C_1(j), \quad j^n=11n+22\theta^n(1+\theta^tC_1)^n,$

$i^2=21\theta^{\mu_2-1}+22\theta^{\mu_1-t-1}B_3+11\delta^{\mu_1-t-1}B_3+12\delta^{\mu_1-\mu_2-t-2}B_3$

$=ij^{\mu_2-1}-12\delta^{\mu_1-\mu_2-t-2}B_3+j^{\mu_1-t-1}B_3-22\theta^{\mu_1-t-1}B_3$

$+12\delta^{\mu_1-\mu_2-t-2}B_3+22\theta^{\mu_1-t-1}B_3=ij^{\mu_2-1}+j^{\mu_1-t-1}B_3(j).$

32212. $c_0=1, \quad \mu_1=\mu_2, \quad B=\theta^{\mu_2-t}B_5, \quad s+2=\mu_2.$

$i=210+22\theta^{\mu_2-1}+12\delta^{\mu_1-t}B_5, \quad j=111+221+22\theta^{t+1}C_1,$

$ij^n=21n+12\delta^{\mu_1-t+n}B_5, \quad ji=211+21\theta^{t+1}C_1+12\delta^{\mu_1-t+1}B_5=ij+ij^{t+1}C_1(j),$

$j^n=11n+22\theta^n(1+\theta^tC_1)^n, \quad i^2=22\theta^{\mu_1-t}B_5+21\theta^{\mu_2-1}+11\delta^{\mu_1-t}B_5,$

$i^2=ij^{\mu_2-1}+j^{\mu_1-t}B_5(j).$

32221. $c_0=-1$, μ_2-1 is even, μ_1-t-1 is even, $B=\theta^{\mu_2-1}b$, $s+2=\mu_2$.

$i=210+22\theta^{\mu_2-1}+12\delta^{\mu_1-1}b, \quad j=111+221(-1)+22\theta^{t+1}C_1, \quad ij=211,$

$ji=211(-1)+21\theta^{t+1}C_1=-ij+ij^{t+1}C_1(j), \quad j^n=11n+22\theta^n(-1+\theta^tC_1)^n,$

$i^2=21\theta^{\mu_2-1}+22\theta^{\mu_1-1}b+11\delta^{\mu_1-1}b$

$=ij^{\mu_2-1}+j^{\mu_1-1}b-22\theta^{\mu_1-1}b(-1)^{\mu_1-1}+22\theta^{\mu_1-1}b.$

Since $\theta^{\mu_1-1}=0$, unless $\mu_1=\mu_2$, these last two terms vanish separately. If $\mu_1=\mu_2$ they destroy each other since $(-1)^{\mu_2-1}=1$.

32222. $c_0=-1$, μ_2-1 is even, μ_1-t-1 is odd, $B=0$, $s+2=\mu_2$.

$i=210+22\theta^{\mu_2-1}, \quad j=111+221(-1)+22\theta^{t+1}C_1, \quad ij=211,$

$ji=211(-1)+21\theta^{t+1}C_1=-ij+ij^{t+1}C_1(j), \quad i^2=21\theta^{\mu_2-1}=ij^{\mu_2-1}.$

41. $s+2<\mu_2, \quad \mu_1>\mu_2, \quad c_0=1, \quad t=\mu_2-s-2+r, \quad B=\theta^{s+1-r}B_4.$

$i=210+22\theta^{s+1}+12\delta^{\mu_1-\mu_2+s+1-r}B_4, \quad j=111+221+22\theta^{t+1}C_1=111+22\theta C.$

$ij^n=21n+12\delta^{\mu_1-t+n-1}C^nB_4+22\theta^{s+n+1}C^n,$

$ji=21\theta C+22\theta^{s+2}C+12\delta^{\mu_1-t}B_4=ijC(j)=ij+ij^{t+1}C_1(j).$

$i^2=21\theta^{s+1}+22\theta^{\mu_1-t-1}B_4+11\delta^{\mu_1-t-1}B_4+12\delta^{\mu_1-t+s}B_4+22\theta^{2s+2}$

$=ij^{s+1}+j^{\mu_1-t-1}B_4.$

42. $s+2<\mu_2, \quad \mu_1=\mu_2, \quad c_0=1, \quad t=\mu_2-s-2+r, \quad B=\theta^{s+2-r}B_6.$

$i=210+22\theta^{s+1}+12\delta^{s+2-r}B_6, \quad j=111+22\theta C,$

$ij^n=21n+22\theta^{s+n+1}C^n+12\delta^{s+n+2-r}B_6C^n,$

$ji=21\theta C+22\theta^{s+2}C+12\delta^{s+2-r}B_6=ijC(j)=ij+ij^{t+1}C_1(j),$

$ij^2=21\theta^{s+1}+22\theta^{2s+2}+22\theta^{s+2-r}B_6+11\delta^{s+2-r}B_6+12\delta^{2s+3-r}B_6$

$=ij^{s+1}+j^{s+2-r}B_6(j).$

Type	$ji=$	$i^2=$	
1	$ij+ij^{t+1}C_1(j)$,	$ij^{\mu_2-1}+j^{\mu_1-t-1}B_3(j)$,	$\mu_1>\mu_2$,
2	$ij+ij^{t+1}C_1(j)$,	$ij^{\mu_2-1}+j^{\mu_1-t}B_5(j)$,	$\mu_1=\mu_2$,
3	$ij+ij^{t+1}C_1(j)$,	$ij^{s+1}+j^{\mu_1-t-1}B_4(j)$,	$\mu_1>\mu_2$,
4	$ij+ij^{t+1}C_1(j)$,	$ij^{s+1}+j^{\mu_2-t}B_6(j)$,	$\mu_1=\mu_2$,
5	$c_0ij+ij^{t+1}C_1(j)$,	ij^{μ_2-1},	$c_0^2\neq 1$, $\mu_1=\mu_2$,
6	$c_0ij+ij^{t+1}C_1(j)$,	$ij^{\mu_2-1}+j^{\mu_1-1}b$,	$c_0^2\neq 1$, $\mu_1>\mu_2$,
7	$c_0ij+ij^{t+1}C_1(j)$,	$ij^{\mu_2-1}+j^{\mu_1-1}b$,	$c_0^2\neq 1$, $c_0^{\mu_2-1}=1$,
8	$-ij+ij^{t+1}C_1(j)$,	ij^{μ_2-1},	μ_1-t is even, μ_2 is odd,
9	$-ij+ij^{t+1}C_1(j)$,	$ij^{\mu_2-1}+j^{\mu_1-1}b$,	μ_1-t is odd, μ_2 is odd,
10	$-ij+ij^{t+1}C_1(j)$,	$ij^{\mu_2-1}+j^{\mu_1-\mu_2+v}B_2(j)$,	$v\geq 2\mu_2-\mu_1$, μ_1-v is odd,
11	$-ij+ij^{t+1}C_1(j)$,	$ij^{\mu_2-1}+j^{\mu_1-\mu_2+v}B_2(j)$,	$v\geq 2\mu_2-\mu_1-1$, μ_1-v is even.

Case III_2. $c_0=0$.

$C=\theta^tC_1(\theta)$, $0<t\leq\mu_2-2$. We go back to equations (A) which now become,

(1) $\theta^{t+2}A[1-C(\theta C)]=0$, for which we have either

1. $t=\mu_2-2$, $C=\theta^{\mu_2-2}c$, where c is written for c_{μ_2-2}, or
2. $A=\theta^{\mu_2-t-2}A_1$.

(2) $\delta^{\mu_1-\mu_2+1}B=0$. $\therefore$ $B=\delta^{\mu_2-1}b$, where b may be zero.

(3) $\delta^{\mu_1-\mu_2+1}B[A-C\cdot A(\theta C)]=0$. This is satisfied by the result of (2).

We now use the results of equations (1) and (2) in (4). For 1 $t=\mu_2-2$, $C=\theta^{\mu_2-2}c$, $B=\delta^{\mu_2-1}b$, equation (4) becomes

(4) $\theta^2A[A-\theta^{\mu_2-2}cA(\theta C)]+\theta^{\mu_1-1}b[1-C^{\mu_1-1}]=0$, $\theta^2A^2+\theta^{\mu_1-1}b=0$.

The three possible solutions of this are

11, $\mu_1>\mu_2$, $A=\theta^{m-1}A_2$ where $m=\mu_2/2$ if μ_2 is even,

$$m=\frac{\mu_2-1}{2} \text{ if } \mu_2 \text{ is odd,}$$

12, $\mu_1=\mu_2=\mu$, $A=\theta^{\frac{\mu-2}{2}}A_3$, $B=0$, μ is even.

13, $\mu_1=\mu_2=\mu$, $A=\theta^{\frac{\mu-3}{2}}A_4$, $B=\delta^{\mu_2-1}b$, $a_{40}^2+b=0$, μ is odd.

For 2 $C=\theta^tC_1$, $B=\delta^{\mu_2-1}b$, $A=\theta^{\mu_2-t-2}A_1$, (4) becomes

$$(4)\qquad \theta^{2\mu_2-2t-2}A_1[A_1-C^{\mu_2-t-1}A_1(\theta C)]+\theta^{\mu_1-1}b=0,$$
$$\theta^{2(\mu_2-t-1)}A_1^2-\theta^{(\mu_2-t-1)t+2\mu_2-2t-2}A_1C_1A_1(\theta C)+\theta^{\mu_1-1}b=0.$$

This reduces to

$$\theta^{2(\mu_2-t-1)}A_1^2+\theta^{\mu_1-1}b=0,$$

because $(\mu_2-t-1)t+2\mu_2-2t-2\geq\mu_2$, *i. e.*, $(t+2)\mu_2-(t^2+3t+2)\geq\mu_2$, or $\mu_2-t-2\geq 0$, $t\leq\mu_2-2$, which is true for this case.

Now the possible solutions of

$$\theta^{2(\mu_2-t-1)}A_1^2+\theta^{\mu_1-1}b=0$$

are 21, $\mu_1>\mu_2$, which gives rise to

211, $2t\leq\mu_2-2$,

and 212, $2t>\mu_2-2$, $A_1=\theta^{t+1-m}A_2$, $m=\frac{\mu_2}{2}$ if μ_2 is even,

$m=\frac{\mu_2+1}{2}$ if μ_2 is odd,

and 22, $\mu_1=\mu_2=\mu$, which gives rise to the following:

221, $2t\leq\mu_2-2$, $b=0$,

222, $2t>\mu_2-2$, $A_1=\theta^{t+1-\frac{\mu}{2}}A_2$, $b=0$, μ is even,

223, $2t>\mu-2$, $A_1=\theta^{t+1-\frac{\mu-1}{2}}A_2'$, $a_{20}^2+b=0$, μ is odd.

Therefore we have the following eight types:

11. $\mu_1>\mu_2$, $A=\theta^{m-1}A_2$, $B=\theta^{\mu_2-1}b$, $C=\theta^{\mu_2-2}c$, $m=\frac{\mu_2}{2}$ if μ_2 is even.

$m=\frac{\mu_2-1}{2}$ if μ_2 is odd.

12. $\mu_1=\mu_2=\mu$ is even, $A=\theta^{\frac{\mu-2}{2}}A_3$, $B=0$, $C=\theta^{\mu-2}c$.

13. $\mu_1=\mu_2=\mu$ is odd, $A=\theta^{\frac{\mu-3}{2}}A_4$, $B=\theta^{\mu-1}b$, $C=\theta^{\mu-2}c$, $a_{40}^2+b=0$.

211. $\mu_1>\mu_2$, $2t\leq\mu_2-2$, $A=\theta^{\mu_2-t-2}A_1$, $B=\delta^{\mu_2-1}b$, $C=\theta^tC_1$.

212. $\mu_1>\mu_2$, $2t\leq\mu_2-2$, $A=\theta^{\mu_2-m-1}A_2$, $B=\delta^{\mu_2-1}b$, $C=\theta^tC_1$,

$m=\frac{\mu_2}{2}$ if μ_2 is even, $m=\frac{\mu_2+1}{2}$ if μ_2 is odd.

221. $\mu_1=\mu_2=\mu$, $2t\leq\mu_2-2$, $A=\theta^{\mu_2-t-2}A_1$, $B=0$, $C=\theta^tC_1$.

222. $\mu_1=\mu_2=\mu$ is even, $2t>\mu-2$, $A=\theta^{\frac{\mu}{2}-1}A_2$, $B=0$, $C=\theta^tC_1$.

223. $\mu_1=\mu_2=\mu$ is odd, $2t>\mu-2$, $A=\theta^{\frac{\mu-1}{2}-1}A_2'$, $B=\delta^{\mu-1}b$, $C=\theta^tC_1$, $a_{20}^2+b=0$.

These eight written out give the following:

11. $\mu_1>\mu_2$, $A=\theta^{m-1}A_2$, $B=\delta^{\mu_2-1}b$, $C=\theta^{\mu_2-2}C_1$, $m=\frac{\mu_2}{2}$ if μ_2 is even,

$m=\frac{\mu_2-1}{2}$ if μ_2 is odd.

$i=210+22\theta^mA_2+12\delta^{\mu_1-1}b$, $j=111+22\theta^{\mu_2-1}c$.

Hence $ij=211$, $ji=21\theta^{\mu_2-1}c=ij^{\mu_2-1}c$, $i^2=21\theta^mA_2+11\delta^{\mu_1-1}b=ij^mA_2(j)+j^{\mu_1-1}b$.

12. $\mu_1=\mu_2=\mu$ is even, $A=\theta^{\frac{\mu-2}{2}}A_3$, $B=0$, $C=\theta^{\mu-2}c$.

$i=210+22\theta^{\frac{\mu}{2}}A_3$, $j=111+22\theta^{\mu-1}c$, $ij=211$,

$ji=21\theta^{\mu-1}c=ij^{\mu-1}c$, $i^2=21\theta^{\frac{\mu}{2}}A_3=ij^{\frac{\mu}{2}}A_3(j)$.

13. $\mu_1=\mu_2=\mu$ is odd, $A=\theta^{\frac{\mu-3}{2}}A_4$, $B=\delta^{\mu_1-1}b$, $C=\theta^{\mu-2}c$, $a_{40}^2+b=0$.
$i=210+22\theta^{\frac{\mu-1}{2}}A_4+12\delta^{\mu-1}b$, $j=111+22\theta^{\mu-1}c$, $ij=211$,
$ji=21\theta^{\mu-1}c=ij^{\mu-1}c$,
$i^2=21\theta^{\frac{\mu-1}{2}}A_4+22\theta^{\mu-1}a_{40}^2+22\theta^{\mu-1}b+11\delta^{\mu-1}b=ij^{\frac{\mu-1}{2}}A_4(j)+j^{\mu-1}b$.

211. $\mu_1>\mu_2$, $2t\leq\mu_2-2$, $A=\theta^{\mu_2-t-2}A_1$, $B=\delta^{\mu_2-1}b$, $C=\theta^tC_1$.
$i=210+22\theta^{\mu_2-t-1}A_1+12\delta^{\mu_1-1}b$, $j=111+22\theta^{t+1}C_1$, $ij=211$,
$ji=21\theta^{t+1}C_1=ij^{t+1}C_1(j)$, $i^2=21\theta^{\mu_2-t-1}A_1+11\delta^{\mu_1-1}b=ij^{\mu_2-t-1}A_1(j)+j^{\mu_1-1}b$.

212. $\mu_1>\mu_2$, $2t\leq\mu_2-2$, $A=\theta^{\mu_2-m-1}A_2$, $B=\delta^{\mu_2-1}b$, $C=\theta^tC_1$,
$m=\frac{\mu_2}{2}$ if μ_2 is even, $m=\frac{\mu_2+1}{2}$ if μ_2 is odd.
$i=210+22\theta^{\mu_2-m}A_2+12\delta^{\mu_1-1}b$, $j=111+22\theta^{t+1}C_1$, $ij=211$,
$ji=21\theta^{t+1}C_1=ij^{t+1}C_1(j)$, $i^2=21\theta^{\mu_2-m}A_2+11\delta^{\mu_1-1}b=ij^{\mu_2-m}A_2(j)+j^{\mu_1-1}b$.

221. $\mu_1=\mu_2=\mu$, $2t\leq\mu_1-2$, $A=\theta^{\mu-t-2}A_1$, $B=0$, $C=\theta^tC_1$.
$i=210+22\theta^{\mu-t-1}A_1$, $j=111+22\theta^{t+1}C_1$, $ij=211$,
$ji=21\theta^{t+1}C_1=ij^{t+1}C_1(j)$, $i^2=21\theta^{\mu-t-1}A_1=ij^{\mu-t-1}A_1(j)$.

222. μ is even, $2t>\mu-2$, $A=\theta^{\frac{\mu}{2}-1}A_2$, $B=0$, $C=\theta^tC_1$.
$i=210+22\theta^{\frac{\mu}{2}}A_2$, $j=111+22\theta^{t+1}C_1$, $ij=211$,
$ji=21\theta^{t+1}C_1=ij^{t+1}C_1(j)$, $i^2=21\theta^{\frac{\mu}{2}}A_2=ij^{\frac{\mu}{2}}A_2(j)$.

223. μ is odd, $2t>\mu-2$, $A=\theta^{\frac{\mu-1}{2}-1}A_2'$, $B=\delta^{\mu-1}b$, $C=\theta^tC_1$, $a_{20}'^2+b=0$.
$i=210+22\theta^{\frac{\mu-1}{2}}A_2'+12\delta^{\mu-1}b$, $j=111+22\theta^{t+1}C_1$, $ij=211$,
$ji=21\theta^{t+1}C_1=ij^{t+1}C_1(j)$,
$i^2=21\theta^{\frac{\mu-1}{2}}A_2'+11\delta^{\mu-1}b+22\theta^{\mu-1}b+22\theta^{\mu-1}a_{20}'^2=ij^{\frac{\mu-1}{2}}A_2'(j)+j^{\mu-1}b$.

Type	$ji=$	$i^2=$	
1	$ij^{\mu_2-1}c$,	$ij^mA_2(j)+j^{\mu_1-1}b$,	$\mu_1>\mu_2$,
2	$ij^{\mu-1}c$,	$ij^{\frac{\mu}{2}}A_3(j)$,	$\mu_1=\mu_2=\mu$ is even,
3	$ij^{\mu-1}c$,	$ij^{\frac{\mu-1}{2}}A_4(j)+bj^{\mu-1}$,	μ is odd, $b+a_{40}^2=0$,
4	$ij^{t+1}C_1(j)$,	$ij^{\mu_2-t-1}A_1(j)+bj^{\mu_1-1}$,	
5	$ij^{t+1}C_1(j)$,	$ij^{\mu_2-m}A_2(j)+bj^{\mu_1-1}$,	$m=\frac{\mu_2}{2}$ if μ_2 is even, $m=\frac{\mu_2+1}{2}$ if μ_2 is odd.
6	$ij^{t+1}C_1(j)$,	$ij^{\mu-t-1}A_1(j)$,	$2t\leq\mu-2$,
7	$ij^{t+1}C_1(j)$,	$ij^{\frac{\mu}{2}}A_2(j)$,	$2t>\mu-2$, μ is even,
8	$ij^{t+1}C_1(j)$,	$ij^{\frac{\mu-1}{2}}A_2'(j)+j^{\mu-1}b$,	$b+a_{20}'^2=0$, $2t>\mu-2$, μ is odd.

VII. *Complete List of the Types Arranged According to the Form of the Product ji, with Sufficient Conditions to Distinguish Each.*

Class	Type	$ji=$	$i^2=$	Conditions
I	1	ij,	$i'^2=j^{\mu_1-\mu_2}B(j)$,	$E=0$.
II	2	$ij+j^{\mu_2+1}E_1(j)$,	$i'^2=j^{\mu_2}B(j)$,	$\mu_1-\mu_2=\mu_2,\ E_1\neq 0$,
	3	$ij+j^{\mu_2+\alpha}E(j)$,	$i'^2=j^{\mu_2+\alpha}B(j)$,	$\mu_1-\mu_2=\mu_2+\alpha,\ \alpha>0,\ E\neq 0$,
	4	$ij+j^{\mu_1-\mu_2+v}E_3(j)$,	$i'^2=j^{\mu_1-\mu_2}B(j)$,	$\mu_1-\mu_2<\mu_2,\ v>2\mu_2-\mu_1,\ E_3\neq 0$.
III	5	$ij+ij^{t+1}C_1(j)$,	$j^{\mu_1-t}B_2(j)$,	$\mu_1=\mu_2$,
	6	$ij+ij^{t+1}C_1(j)$,	$j^{\mu_1-t-1}B_1(j)$,	$\mu_1>\mu_2$,
	7	$ij+ij^{t+1}C_1(j)$,	$ij^{\mu_2-1}+j^{\mu_1-t-1}B_3(j)$,	$\mu_1>\mu_2$,
	8	$ij+ij^{t+1}C_1(j)$,	$ij^{\mu_2-1}+j^{\mu_1-t}B_5(j)$,	$\mu_1=\mu_2$,
	9	$ij+ij^{t+1}C_1(j)$,	$ij^{s+1}+j^{\mu_1-t-1}B_4(j)$,	$\mu_1>\mu_2$,
	10	$ij+ij^{t+1}C_1(j)$,	$ij^{s+1}+j^{\mu_2-t}B_6(j)$,	$\mu_1=\mu_2$.
IV	11	$c_0ij+ij^{t+1}C_1(j)$,	0,	$c_0^2\neq 1$,
	12	$c_0ij+ij^{t+1}C_1(j)$,	$j^{\mu_1-1}b$,	$c_0^2\neq 1,\ b\neq 0$,
	13	$c_0ij+ij^{t+1}C_1(j)$,	$ij^{\mu_2-1}+j^{\mu_1-1}b$,	$c_0^2\neq 1,\ \mu_1>\mu_2$,
	14	$c_0ij+ij^{t+1}C_1(j)$,	ij^{μ_2-1},	$c_0^2\neq 1,\ \mu_1=\mu_2$,
	15	$c_0ij+ij^{t+1}C_1(j)$,	$ij^{\mu_2-1}+j^{\mu_1-1}b$,	$c_0^2\neq 1,\ c_0^{\mu_2-1}=1$.
V	16	$-ij+ij^{t+1}C_1(j)$,	0,	
	17	$-ij+ij^{t+1}C_1(j)$,	$j^{\mu_1-1}b$,	$b\neq 0$,
	18	$-ij+ij^{t+1}C_1(j)$,	ij^{μ_2-1},	μ_1-t is even, μ_2 is odd,
	19	$-ij+ij^{t+1}C_1(j)$,	$j^{\mu_1-t-1}B_1(j)$,	$B_1(j)\neq 0$,
	20	$-ij+ij^{t+1}C_1(j)$,	$ij^{\mu_2-1}+j^{\mu_1-1}b$,	μ_1-t is odd, μ_2 is odd,
	21	$-ij+ij^{t+1}C_1(j)$,	$ij^{\mu_2-1}+j^{\mu_1-\mu_2+v}B_2(j)$,	$v\geq 2\mu_2-\mu_1$, μ_1-v is odd,
	22	$-ij+ij^{t+1}C_1(j)$,	$ij^{\mu_2-1}+j^{\mu_1-\mu_2+v}B_2(j)$,	$v\geq 2\mu_2-\mu_1-1$, μ_1-v is even.
VI	23	$ij^{\mu_2-1}c$,	$ij^{m}A_2(j)+j^{\mu_1-1}b$,	$\mu_1>\mu_2$
	24	$ij^{\mu-1}c$,	$ij^{\frac{\mu}{2}}A_3(j)$,	$\mu_1=\mu_2=\mu$ is even,
	25	$ij^{\mu-1}c$,	$ij^{\frac{\mu-1}{2}}A_4(j)+bj^{\mu-1}$,	$b+a_{40}^2=0$, μ is odd,
	26	$ij^{t+1}C_1(j)$,	$ij^{\mu_2-t-1}A_1(j)+bj^{\mu_1-1}$,	
	27	$ij^{t+1}C_1(j)$,	$ij^{\mu_2-m}A_2(j)+bj^{\mu_1-1}$,	$m=\frac{\mu_2}{2}$ if μ_2 is even,
				$m=\frac{\mu_2+1}{2}$ if μ_2 is odd,
	28	$ij^{t+1}C_1(j)$,	$ij^{\mu-t-1}A_1(j)$,	$2t\leq\mu-2$,
	29	$ij^{t+1}C_1(j)$,	$ij^{\frac{\mu}{2}}A_2(j)$,	$2t>\mu-2$, μ is even,
	30	$ij^{t+1}C_1(j)$,	$ij^{\frac{\mu-1}{2}}A_2'(j)+bj^{\mu-1}$,	$2t>\mu-2$, μ is odd, $b+a_{20}'^2=0$.

These types when arranged according to the form of the product ji fall into six classes.

Class I, made up of type 1, is the only commutative type of the entire set.

Class II, types 2–4, have for the product ji the term ij and powers of j.

Class III, types 5–10, have $ji=ij+ij^{t+1}C_1(j)$ where $t>0$ and $C_1 \neq 0$.

Class IV, types 11–15, differ from those of Class III in that

$$ji=c_0 ij+ij^{t+1}C_1(j) \text{ where } c_0^2 \neq 1.$$

Class V, types 16–22, are essentially different from those of Classes III and IV since $ji=-ij+ij^{t+1}C_1(j)$.

Class VI, types 23–30, have the product ji starting with at least ij^2.

We need now to see that the types of each class are essentially distinct. The conditions given, having arisen in the determination of these types, are sufficient to show this. For example, we might think that type 18 and type 20 would be the same if in type 20 $b=0$. Although this would make the products ji and i^2 have the same expressions for the two types, they would still be distinct, for in 18 μ_1-t is odd and in 20 it is even. Types 13 and 15 have the same form for ji and i^2, but they are essentially distinct because in 13 $\mu_1>\mu_2$, which is not necessarily the case in 15, and furthermore in 15 $c_0^{\mu_2-1}=1$, but the only limitation on c_0 in 13 is $c_0^2 \neq 1$. Similarly it can be shown that the thirty types are all distinct.